

SECTION C — ($3 \times 10 = 30$ marks)

Answer any THREE questions.

16. If $O(G) = P^n$, where P is prime number, prove that $Z(G) \neq (1)$.
17. For isomorphic abelian groups G and G' , show that $G(s)$ and $G(s')$ are isomorphic, if s is an integer.
18. If $T \in A(v)$ has all its characteristic roots in F , show that there is a basis of V in which the matrix of T is triangle.
19. Let $T \in A(v)$ have all its distinct characteristic roots $\lambda_1, \lambda_2, \dots, \lambda_k$ in F . Prove that a basis of V can be found in which matrix T is of the form
$$\begin{pmatrix} J_1 & & \\ & J_2 & \\ & & \ddots \\ & & & J_k \end{pmatrix}$$
 where each $J_i = \begin{pmatrix} B_{i1} & & \\ & \ddots & \\ & & B_{ir_i} \end{pmatrix}$ and where $B_{i1}, \dots, B_{ir_i}$ are basic Jordan belonging to λ_i .
20. (a) Prove that characteristic roots of a hermitian transformation is always real.
(b) If N is normal and if $N(v) = \lambda v$, show that $N^*(v) = (\lambda)(v)$.

APRIL/MAY 2025

23PMA11 — ALGEBRAIC STRUCTURES

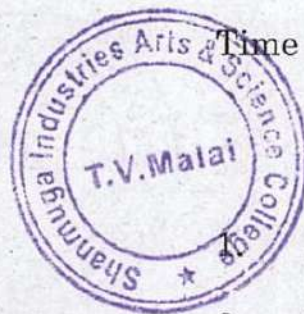
Time : Three hours

Maximum : 75 marks

SECTION A — ($10 \times 2 = 20$ marks)

Answer ALL questions.

1. Prove that $a \in z$ if and only if $N(a) = G$.
2. Define conjugate on an element in G .
3. If $u \in v$ is such that $uT^{n_1-k} = 0$, where $0 < k \leq n$, show that $u = u_0T^k$ for some $u_0 \in v_1$.
4. Define invariant of subspace W and T .
5. Define cyclic of a module over R .
6. Define solvable group.
7. Define companion matrix of $f(x) \in F[x]$.
8. Show that every $T \in A(v)$ satisfies its characteristic polynomial.
9. If A is invertible, prove that $(A \subset A^{-1}) = \text{trc}$.
10. Define unitary transformation.



SECTION B — (5 × 5 = 25 marks)

Answer ALL questions.

11. (a) If $O(G) = p^2$, where p is prime, show that G is abelian.

Or

- (b) Prove that conjugacy is an equivalence relation on G .

12. (a) Show that G is solvable if and only if $G^{(k)} = (e)$ for some integer k .

Or

- (b) Let $G = S_n$, where $n \geq 5$. Show that $G^{(k)}$ for $i = 1, 2, \dots$, contains every 3-cycle S_n .

13. (a) If M is of dimension M is cyclic with respect to T , prove that dimension of $T^k(M) = m - k$.

Or

- (b) If W is invariant under T , and if T , and if T includes $\bar{T} + \frac{V}{W} \rightarrow \frac{V}{W}$ by $\bar{T}(v+w) = T(v) + W$, show that if $p_1(x)$ is the minimal polynomial for $\bar{T}$ and $p(x)$ is that for T , then, $p_1(x) + p(x)$.



14. (a) If $T \in A(v)$ has a minimal polynomial $p(x) = (q(x))^l$, where $q(x)$ is a monic irreducible polynomial in $F[x]$, prove that a basis of v over F can be found in which the matrix of T is of the form

$$\begin{pmatrix} c(q(x))^{e_1} & & \\ & c(q(x))^{e_2} & \\ & & \ddots c(q(x))^{e_r} \end{pmatrix} \text{ where } e = e_1 \geq \dots \geq e_r.$$

Or

- (b) If all distinct roots $\lambda_1, \dots, \lambda_k$ of T lie in F , prove that V can be written as $V = V_1 \oplus \dots \oplus V_k$, where $V_i = \{v \in V \mid (T - \lambda_i)^{l_i}(v) = 0\}$ and where T_i has only one characteristic root λ_i on v_i .

15. (a) For any $T \in A(v)$, prove that T is unitary if and only if $TT^* = I$.

Or

- (b) Prove that $T \geq 0$ if and only if $T = AA^*$ for some $A \in A(v)$.